11) Potential theory

Harmonic functions

Def: $u : U \subset \mathbb{C} \to \mathbb{R}$ is harmonic if u is $C^2(U)$ and $\Delta u = 0$

Rem: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} u = 4 \frac{\partial}{\partial \bar z}\frac{\partial}{\partial z} u$.

Properties: · If $f : U \to \mathbb{C}$ is holomorphic, then $u = \operatorname{Re} f$ is harmonic

· locally (or more generally on simply connected domains), the converse is true:
$\forall u : U \to \mathbb{R}$ harmonic, $\exists f : U \to \mathbb{C}$ holomorphic, $u = \operatorname{Re} f$.
simply connected

· Mean property: $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta$ (caracterizes harmonic functions)

Subharmonic functions.

Def: $u : U \subset \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if:

- u is upper semicontinuous (u.s.c) $u(z_0) \geq \limsup_{z \to z_0} u(z) \} \Leftrightarrow \forall r, \{u < r\}$ is open)
- $\forall\ \overline{D(z,r)}$ (or compact K) $\subset U$, $\forall h : \overline{D(z,r)} \to \mathbb{R}$, C^0 and harmonic on $D(z,R)$,
  $h \geq u$ on $\partial D(z,r) \Rightarrow h \geq u$ on $D(z,r)$.

Properties:

1)- Mean property: $u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta \quad (*)$

u is subharmonic $\Leftrightarrow$ u is usc and $(*)$ holds

2)- if u is C^2, then u is subharmonic $\Leftrightarrow \Delta u \geq 0$

(S.h $\overset{1}{\Leftrightarrow}$ Mean property): directly from the analogous property for ~~holomorphic~~ harmonic maps)
as far as we take the definition with K —

2) See [Klimek, Theorem 2.5.1]

Rem: sometimes, we ask $u \not\equiv -\infty$ on any connected component of U.

Notice that the definition of (harmonic)/subharmonic functions is local, and can be extended to $u: X \to \mathbb{R} \cup \{-\infty\}$, X Riemann Surface

Examples: h is harmonic $\Leftrightarrow$ h and $-h$ are subharmonic

. If $f: U \to \mathbb{C}$ is holomorphic, then $\log f$ is locally (= on small disks / simply connected ~~domains~~) defined, and holomorphic, as long as $f(z) \neq 0$.

Hence $\operatorname{Re} \log f = \log|f|$ is harmonic on $U \cap \{f(z) \neq 0\}$.

If $f(z)=0$, $\log|f(z)| = -\infty$, and $\log|f|$ satisfies the mean property, hence it is subharmonic

(The locus $\{u = -\infty\}$ is called a "polar" set. $z \,|\, u(z) = -\infty$ is called a singularity of u)

Max. principle: $u: U \to \mathbb{R}^{\cup\{-\infty\}}$ subharmonic. Suppose that $a \in U$ is a local maximum point of u. Then $u \equiv u(a)$ locally at a.

If a is a global maximum and U is connected, then $u \equiv u(a)$ on U.

Proof: local statement: $D(a,r)$. global: K any compact $a \in \mathring{K}$.

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta \underset{\substack{\uparrow \\ a \text{ local max}}}{\leq} u(a) \qquad \forall r<<1 \Rightarrow u(z) \equiv u(a) \underset{\substack{\uparrow \\ \text{almost every}}}{\forall} z \; |z-a|=r$$

but by u.s.c, the points $\tilde{z}$ where $u(z) \neq u(a)$ (maybe) satisfy $u(z) \geq u(a)$, and $u(z) \leq u(a)$ since a is a max. point. $\square$

Limits:

. If $(u_n)_n$ is a decreasing sequence of subharmonic functions, then $u = \lim_{n\to\infty} u_n$ is subharmonic

- If $(u_\alpha)_{\alpha\in A}$ is a ~~fam~~ (locally bounded) family of subharmonic functions, and $u = \sup_\alpha u_\alpha$ is u.s.c, then u is subharmonic.

Rem: the decreasing limit of u.s.c. functions is u.s.c

Since $\{u_n < c\}$ is an increasing sequence of opens, and $\{u<c\} = \bigcup_n \{u_n < c\}$.

The increasing limit of u.s.c functions is not necessarily u.s.c!

Ex: $u_n(z) = \begin{cases} 1 & |z| \geq \frac{1}{n} \\ n|z| & |z| \leq n \end{cases}$ is continuous, $u_n(z) \nearrow \underset{\sup u_n}{u}(z) = \begin{cases} 1 & z \neq 0 \\ 0 & z = 0 \end{cases}$.

Proof of subharmonicity: $u_n \searrow u$

$$u(z) = \lim_n u_n(z) \leq \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(z+re^{i\theta})d\theta \underset{\substack{\uparrow \\ \text{monotone convergence thm.}}}{=} \frac{1}{2\pi}\int_0^{2\pi} \lim_n u_n(z+re^{i\theta})d\theta$$

$u = \sup_\alpha u_\alpha$: $u_\alpha(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u_\alpha(z+re^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$ $\forall \alpha \in A$ $\Rightarrow$

$\Rightarrow u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$. well defined because u is u.s.c.

Rem: more generally, one can consider the upper semicontinuous regularization u^* of u, defined by $u^*(z) = \underset{\substack{\uparrow \\ =\inf}}{\lim_{\varepsilon\to 0}} \sup_{B(z,\varepsilon)} u = \limsup_{z'\to z} u(z')$.

u^* is the smallest u.s.c. function $\geq u$, and coincides with u almost everywhere.

• If u_α are all subharmonic, then u^* is subharmonic.

Idea: • Choquet's lemma! May assume A is countable.

• this implies $\sup u_\alpha$ is measurable and can apply integration theorems.

Theorem: If U is connected and $u: U \to \mathbb{R}\cup\{-\infty\}$ is subharmonic with $u \not\equiv -\infty$, then $u \in L^1_{loc}(U)$.

Proof: Let $W = \{z_0 \in U \mid u \text{ is } L^1_{loc} \text{ at } z_0, \text{ i.e. } \exists D(z_0,r), \int_{D(z_0,r)} u(z)\,d\mathcal{L} \in \mathbb{R}.\}$

By definition, W is open, and $u > -\infty$ almost everywhere in W. Notice that if $u(z_0) > -\infty$, then $\int_{D(z_0,r)} u(z)\,d\mathcal{L}(z) = \int_0^r\int_0^{2\pi} u(z_0+\rho e^{i\theta})\rho\,d\rho\,d\theta \geq u(z_0)\cdot 2\pi\int_0^r \rho\,d\rho \geq \pi r^2\cdot u(z_0) > -\infty$

while $u(z)$ is bounded above on $\overline{D(z_0,r)}$ by u.s.c. of u, hence $\{u>-\infty\} \subset W$.

If $z \in \partial W \cap U$, let ~~$z \in W$~~ $r := d(z, U^c) > 0$, and let a be such that $a \in W$; $u(a) > -\infty$; $|z - a| < r$. (ok because $u = -\infty$ has measure 0 on W).
Then $z \in D(a,r) \subset W$ $\therefore$ and W is closed.
Being W open and closed, and $\neq \emptyset$ because $\{u > -\infty\} \subset W$, and $u \not\equiv -\infty$, we get $W = U$ and $u \in L^1_{loc}(U)$. $\square$

Subharmonic functions and positive measures.

Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function. If $u \in \mathcal{C}^2$, then $\Delta u \geq 0$ induces a positive measure μ_u on U, by setting:

$$\mu_u(A) = \int_A \Delta u(z) \, d\mathcal{L}(z) \geq 0 \qquad \mathcal{L} \text{ Lebesgue measure on } U.$$

In other terms, μ_u is the absolute continuous measure (w.r.t. $\mathcal{L}$) of density Δu.
It turns out that the same happens when u is only u.s.c.; by taking the Laplacian of u in the sense of distributions.

Distributions: the idea is to consider how objects (e.g. functions, measures) act on a set $\mathcal{D}(U)$ of regular functions (called test functions).
This allows to extend the concept of functions and measure on a new concept, as well as define derivatives "in the sense of distributions" when the maps are not regular enough to admit derivatives in the classical sense.

We set: $\mathcal{D}(U) = \mathcal{C}^\infty_0(U) = \mathcal{C}^\infty$ maps $\varphi : U \to \mathbb{R}$ with compact support.
$(\subset \mathbb{R}^N)$

$\mathcal{D}(U)$ admits a topology, by setting $\varphi_n \to \varphi$ if $\bigcup \operatorname{supp} \varphi_n \subset K \subset U$ for some compact K, and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly for any multiindex α.

Def: a distribution T is a continuous linear functional $T : \mathcal{D}(U) \to \mathbb{R}$ (or $\mathbb{C}$).

We denote: $T(\varphi) = \langle T, \varphi \rangle$.

Examples:

- If $u: U \to \mathbb{R} \cup \{-\infty\}$ is $\in L^1_{loc}(U)$, then u induces a distribution:

$$\langle u, \varphi \rangle = \int_U u(z)\varphi(z)\, d\mathcal{L}(z). \quad \forall \varphi \in \mathcal{D}(U).$$

- If μ is a measure ~~f~~ on U, then it induces a distribution

$$\langle \mu, \varphi \rangle = \int \varphi(z)\, d\mu(z).$$

If $\mu \ll \mathcal{L}$ with density u, then $T_\mu = T_u$

Rem: not all distributions come from measures. (eg: δ_0' derivative of Dirac at 0)

Derivative: If T is a distribution, and ~~if~~ ∂ a derivation (e.g. $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial z}$),

we define ∂T as: $\langle \partial T, \varphi \rangle = -\langle T, \partial \varphi \rangle$,

This comes from the integration by parts in the case of functions: ($U \subset \mathbb{R}$)

$$\langle \tfrac{\partial u}{\partial x}, \varphi \rangle = \int \frac{\partial u}{\partial x}(x)\varphi(x)\,dx = \underbrace{[u(x)\cdot\varphi(x)]_{\partial U}}_{0} - \int u(x)\cdot\frac{\partial \varphi}{\partial x}(x)\,dx.$$

In particular: $\langle \Delta u, \varphi \rangle = \langle u, \Delta\varphi \rangle = \int u(z)\Delta\varphi(z)\, d\mathcal{L}(z).$

Theorem: Let u be a subharmonic function on $U \subset \mathbb{C}$. Then $\forall \varphi \in \mathcal{D}(U)$,

$$\int u(z)\Delta\varphi(z)\,d\mathcal{L}(z) \geq 0.$$

Proof: Use convolutions to reduce to the smooth case:

$u * v(x) = \int u(x-y)v(y)\,d\mathcal{L}(y)$ well defined for $u, v \in \mathcal{L}^1$, or

for $u \in L^1_{loc}$, $v \in L^1$ with compact support

Apply to a sequence of "smoothing kernels": $h(t) = \begin{cases} e^{-\frac{1}{t}} & t>0 \\ 0 & t \leq 0 \end{cases} \in \mathcal{C}^\infty(\mathbb{R})$,

$\chi(z) = C^{-1} \cdot h(1-|z|^2) \in \mathcal{C}^\infty(\mathbb{C})$, $\operatorname{supp}\chi = \overline{D(0,1)}$, $\int_{\mathbb{C}} \chi(z)\,d\mathcal{L}(z) = 1$

normalising constant: $C = \int_{D(0,1)} h(1-|z|^2)\,d\mathcal{L}(z).$

Take $\chi_\varepsilon(z) = \frac{1}{\varepsilon^2}\cdot\chi\left(\frac{z}{\varepsilon}\right)$. (χ_ε) are called standard smoothing kernels.

$\int \chi_\varepsilon(z)\, d\mathcal{L}(z) = 1$, $\operatorname{supp} \chi_\varepsilon = D(0,\varepsilon)$

$u_\varepsilon := u * \chi_\varepsilon \in C^\infty(U_\varepsilon = \{z \in U, d(z, U^c) > \varepsilon\})$

$u_\varepsilon \to u$ uniformly on compacts, and on L^1_{loc}

The converse is also true:

Theorem: If $u : U \to \mathbb{R}\cup\{-\infty\}$ is $L^1_{loc}(U)$ and $\Delta u \geq 0$ in the sense of distributions, then $\exists!\, \tilde{u}$ subharmonic and such that $u \equiv \tilde{u}$ almost everywhere

Idea of the proof:

- Smooth case: by the fact that $\Delta u \geq 0 \iff u$ subharmonic.
- Use smoothing kernels getting u_ε subharmonic defined on U_ε, and setting $\tilde{u}(z) = \lim_{\varepsilon\to 0} u_\varepsilon(z)$.

Rem: not all measures come from this construction. (e.g. the dirac measure)
If it does, $\mu = \Delta u$ (in the sense of distributions) and we say that u is a potential of μ.

Rem: in terms of currents ($\sim$ forms with distribution coefficients), we would have $\mu = i\partial\bar{\partial} u$. The factor i comes from positive forms formalism; μ is a (1-1) form, which is a measure since we are working on $\mathbb{C}$.

Applications to polynomial dynamics

Let $f \in \mathbb{C}[z]$, $\deg f = d \geq 2$. (In chapter 7) we constructed the Green function $G_f : \mathbb{C} \to \mathbb{R}_+$, defined as $G_f(z) = \begin{cases} 0 & z \in K(f) \\ \log|\Phi(z)| = ~~\ldots~~ & z \in \mathbb{C} \setminus K = A_\infty \end{cases}$

where Φ is the Böttcher coordinate of ∞.

G can be rewritten as: $G_f(z) = \lim\limits_{n \to \infty} \frac{1}{d^n} \log^+|f^{\circ n}(z)|$, where $\log^+ = \log \vee 0$
$\max(\log, 0)$

(in fact, if $z \in K$, then $f^{\circ n}(z)$ is bounded, and $\lim \frac{1}{d^n} \log^+|f^{\circ n}(z)| \to 0$)

$\log$ is subharmonic, as is $\log^+$ (the max of subharmonic functions is subharmonic) (more generally, if h is convex and (u_j) subharmonic, then $h(u_1, \ldots, u_N)$ is subharmonic, proved by ~~taking~~ showing the mean property)

Hence $\frac{1}{d^n} \log^+|f^n(z)|$ is also subharmonic (and continuous)

$g_n = \frac{1}{d^n} \log^+|f^n(z)|$

The limit is uniform on compacts. (uniform convergence: estimate of $|g_{n+1} - g_n|$ similar to $|G_{n+1} - G_n|$ in the classical case)

It follows that G_f is continuous and subharmonic

Another way to see it: by ~~...~~ Φ is holomorphic in a nbhd of ∞, and $\Phi(z) = (\Phi(f^n(z)))^{\frac{1}{d^n}}$ is locally well defined since ∞ is totally invariant.

so $\log|\Phi|$ is harmonic on $\mathbb{C} \setminus K$, with $\log|\Phi| > 0$.

Can be also shown that $|\Phi(z)| \to 1$ when $z \to \partial K$, hence G is continuous

Finally, it clearly satisfies the mean property on K, and it is subharmonic.

Its Laplacian (in the sense of distributions) induces a measure μ_f.

~~The measure~~ Definition: The measure $\mu_f = \Delta G_f$ is called the Brolin measure associated to $f \in \mathbb{C}[z]$

Other names: equilibrium measure, Green measure

Theorem: Let $f \in \mathbb{C}[z]$ be a polynomial map, and μ_f its equilibrium measure. Then: $\operatorname{supp} \mu_f = J_f$, and μ_f is ~~not~~ a non-atomic measure.

Proof: Since G_f is harmonic on $\mathbb{C} \setminus J_f = A_\infty \cup \mathring{K}_f$, we have that $\Delta G_f|_{\mathbb{C}\setminus J_f} \equiv 0$ and $\operatorname{supp} \mu_f \subseteq J_f$.

Suppose now that $z \in J_f$, and $\exists\, U \ni z$ nbhd, $\mu_f \equiv 0$ on U. This would imply that G_f is harmonic on U ($\mu_f \equiv 0 \Rightarrow \Delta G_f \equiv 0$ almost everywhere $\Rightarrow$ everywhere since G_f is continuous).
So $-G_f$ is also harmonic and satisfies the mean value property (nbd $\Rightarrow$). Being $-G_f \leq 0$, and $-G_f = 0$ on K_f, we would get $-G_f \equiv G_f \equiv 0$ on a neighborhood U' of z. Since $U' \cap A_\infty \neq \emptyset$, we get a contradiction.
Hence $\operatorname{supp}(\mu_f) = J_f$.

We now prove that μ_f is non-atomic.
Let $z_0 \in J_f$, and a disk $D(z_0, \varepsilon)$. We want to show that $\mu_f(D(z_0,\varepsilon)) \xrightarrow[\varepsilon \to 0]{} 0$.
Let $\psi : \mathbb{C} \to [0,\infty)$ be any C^∞ function, with compact support in $D(0,2)$ and such that $\psi \equiv 1$ on $\overline{D(0,1)}$.
Set $\psi_\varepsilon(z) = \psi\left(\frac{z-z_0}{\varepsilon}\right)$ which has support on $D(z_0, 2\varepsilon)$, and $\psi_\varepsilon \equiv 1$ on $\overline{D(z_0,\varepsilon)}$.
Notice that $\Delta\psi_\varepsilon(z) = \frac{1}{\varepsilon^2}\Delta\psi\left(\frac{z-z_0}{\varepsilon}\right)$. We take

$$\int_{\mathbb{C}} |\Delta\psi_\varepsilon(z)|\, d\mathcal{L}(z) = \frac{1}{\varepsilon^2}\int \left|\Delta\psi\left(\tfrac{z-z_0}{\varepsilon}\right)\right| d\mathcal{L}(z) = \int_{\mathbb{C}} |\Delta\psi(w)|\, d\mathcal{L}(w) \quad (<\infty).$$

Hence: $$\mu_f(D(z_0,\varepsilon)) \leq \int_{\mathbb{C}} \psi_\varepsilon \, d\mu_f = \int_{\mathbb{C}} G_f(z) \cdot \Delta\psi_\varepsilon(z)\, d\mathcal{L}(z) \leq \sup_{D(z_0,2\varepsilon)} |G_f| \cdot \int_{\mathbb{C}} |\Delta\psi_\varepsilon(z)|\, d\mathcal{L}(z)$$

$$= \underbrace{\sup_{D(z_0,2\varepsilon)} |G_f|}_{\xrightarrow[\varepsilon\to 0]{} 0 \text{ by continuity}} \cdot \underbrace{\int |\Delta\psi|\, d\mathcal{L}}_{<\infty}$$

□

Green functions and equilibrium measure for rational maps

The idea (Lyubich) to deal with the Rational case is to lift the situation to homogeneous coordinates in $\mathbb{C}^2$.

In fact, recall that $\hat{\mathbb{C}} \simeq \mathbb{C}\mathbb{P}^1$, where
(also denoted $\mathbb{P}^1_{\mathbb{C}}$, or just $\mathbb{P}^1$)

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 \setminus \{0\}}{\sim}, \quad \underset{z}{(z_0, z_1)} \sim \underset{w}{(w_0, w_1)} \iff \exists \lambda \in \mathbb{C}^*,\ z = \lambda w.$$

An equivalence class is denoted $[z_0 : z_1]$

$\mathbb{P}^1$ is naturally isomorphic to $\hat{\mathbb{C}}$. The chart at 0 corresponds to $z = \frac{z_1}{z_0}$ on $\{z_0 \neq 0\}$, which corresponds to points $[1 : z_1]$. Similarly at ∞ we have the local parameter $w = \frac{z_0}{z_1} \left(\frac{1}{z}\right)$ on $\{z_1 \neq 0\} = \{[z_0 : 1]\}$. Denote by $pr : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ the natural projection.

Proposition Any rational map $f = \frac{P}{Q} \in \mathbb{C}(z)$ of degree $d (\geq 1)$ lifts to a polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ whose coordinates are homogeneous polynomials of degree d, and $F^{-1}(0) = \{0\}$, so that

$$\begin{array}{ccc} \mathbb{C}^2 \setminus \{0\} & \xrightarrow{F} & \mathbb{C}^2 \setminus \{0\} \\ \downarrow pr & & \downarrow pr \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

Proof. We set $F(z_0, z_1) = \left(z_0^d Q\left(\frac{z_1}{z_0}\right); z_0^d P\left(\frac{z_1}{z_0}\right) \right) =: (F_0; F_1)$

If $P(z) = a_d z^d + \dots + a_0 \rightsquigarrow z_0^d P\left(\frac{z_1}{z_0}\right) = a_d z_1^d + a_{d-1} z_1^{d-1} z_0 + \dots + a_0 z_0^d$, and similarly for Q. Clearly $pr \circ F = f \circ pr$.

Since $d = \deg f = \max(\deg P, \deg Q)$, z_0 does not divide both F_0 and F_1.

Similarly z_1 does not divide both (as $P(0) = Q(0) = 0$).

Assume $z_0 \neq 0$, then $F_1 = F_2 = 0 \iff P(z) = Q(z)$, which ~~happens~~ cannot happen. $\square$

Rem F is unique up to multiplication by a constant $\lambda \in \mathbb{C}^*$.

Rem the homogeneity of F corresponds to the fact that $F(\lambda z) = \lambda^d F(z)$ $\forall \lambda \in \mathbb{C}, z \in \mathbb{C}^2$.

Green function associated to F.

We study the dynamical properties of $F : \mathbb{C}^2 \to \mathbb{C}^2$.

Prop: Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ a homogeneous polynomial map on $\mathbb{C}^2$ of degree $d \geq 2$, non degenerate ($F^{-1}(0) = \{0\}$). Then

(1) $\exists\, C > 1$ s.t. $\frac{1}{C}\|z\|^d \leq \|F(z)\| \leq C\|z\|^d \quad \forall z \in \mathbb{C}^2$

(2) The set $\Omega_F = \{ z \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^n(z) = 0 \}$ is open, bounded, and fibered by disks

($\Omega_F \cap L_{\infty} = \underset{\text{at } 0}{\text{disk}}$ for any line L passing through 0).

(3) F^n converges uniformly towards 0 (resp. ∞, i.e. $\|F^n\| \to \infty$) on any compact of Ω_F (resp. $\mathbb{C}^2 \setminus \overline{\Omega_F}$).

(4) $F(\partial\Omega_F) = \partial\Omega_F$.

Proof: (1) Let $S = \{\|z\| = 1\} \subset \mathbb{C}^2$ be the unit sphere.

By compactness, $\exists C > 1$, $\frac{1}{C} \leq \|F(z)\| \leq C \quad \forall z \in S$. (we use regularity to get $\|F(z)\|$ bounded below)

The estimate follows by homogeneity of F.

(2,3): From 1, by recurrence, we get that $\|F^n(z)\| \leq \left(\frac{1}{2}\right)^n \|z\|$ for $\|z\| \leq r := (2C)^{-\frac{1}{d-1}}$

$\|F^n(z)\| \geq 2^n \|z\|$ " $\|z\| \geq R := (2C)^{\frac{1}{d-1}}$

$$\|F^n(z)\| \leq C\|F^{n-1}(z)\|^d \leq C^{\sum_{j=0}^{n-1} d^j} \|z\|^{d^n} \leq C^{\frac{d^n-1}{d-1}} \|z\|^{d^n} \leq \frac{C^{\frac{d^n-1}{d-1}}}{(2C)^{\frac{d^n-1}{d-1}}} \|z\| \leq \left(\frac{1}{2}\right)^{\frac{d^n-1}{d-1}} \|z\| \leq \frac{1}{2^n}\|z\|.$$

Hence $\underbrace{\{\|z\| < r\}}_{= \mathbb{B}_r} \subseteq \Omega_F \subseteq \underbrace{\{\|z\| < R\}}_{= \mathbb{B}_R}$

Moreover, $\Omega_F = \bigcup F^{-n}(\mathbb{B}_r)$ is open.

Since $\|F^n\|$ converges uniformly towards 0 on $\mathbb{B}_r$ (as a geometric sequence) the same happens on Ω_F.

Analogously for $F|_{\mathbb{C}^2 \setminus \overline{\Omega}_F}$.

To end the proof of (2), we show that Ω_F is fibered into disks;
this is equivalent to showing that if $z \in \Omega_F$, then $tz \in \Omega_F$ $\forall t, |t|\leq 1$.
This comes from homogeneity, since $\|F^n(tz)\| = |t|^{d^n}\|F^n(z)\|$.

To end the proof of (3) we show that $\mathcal{A}(\infty) := \{z \in \mathbb{C}^2 \mid \|F^n(z)\| \to +\infty\} \overset{?}{=} \mathbb{C}^2 \setminus \overline{\Omega_F}$.

We saw that $\mathbb{C}^2 \setminus B_R \subseteq \mathcal{A}(\infty) \subseteq \mathbb{C}^2 \setminus \overline{\Omega_F}$.

Let $z \notin \mathcal{A}(\infty)$. Hence $\|F^n(z)\| \leq R$ $\forall n$, and then $\|F^n(tz)\| \leq |t|^{d^n} R \to 0$
$\forall t, |t|<1$. Hence $tz \in \Omega_F$ $\forall t, |t|<1$ and $z \in \overline{\Omega_F}$, i.e. $\mathbb{C}^2 \setminus \mathcal{A}(\infty) \subseteq \overline{\Omega_F}$,
which gives $\mathcal{A}(\infty) \supseteq \mathbb{C}^2 \setminus \overline{\Omega_F}$, and we conclude $\mathcal{A}(\infty) = \mathbb{C}^2 \setminus \overline{\Omega_F}$.

(4) Ω_F is totally invariant, as is $\mathbb{C}^2 \setminus \overline{\Omega_F} = \mathcal{A}(\infty)$.
Hence $F(\partial\Omega_F) \subseteq \partial\Omega_F$, is backward invariant, and by surjectivity of F
we get $F(\partial\Omega_F) = \partial\Omega_F$.

$\square$

As for the one-dimensional dynamics, we can construct a Green function

Theorem: Let $F: \mathbb{C}^2 \to \mathbb{C}^2$ be a non-degenerate homogeneous polynomial mapping of degree $d \geq 2$.
For any $z \in \mathbb{C}^2 \setminus \{0\}$, and $n \in \mathbb{N}$, set $G_n(z) := \frac{1}{d^n} \log\|F^n(z)\|$.
Then there exists a function G_F which is:

0) $\mathcal{C}^0$, plurisubharmonic (psh) on $\mathbb{C}^2$, $G_F^{-1}(-\infty) = \{0\}$

1) $\exists K>0$ s.t. $|G_F(z) - G_n(z)| \leq \frac{K}{d^n}$ $\quad \forall z \in \mathbb{C}^2 \setminus \{0\}$.

2) $G_F(tz) = \log|t| + G_F(z)$ $\quad \forall t \in \mathbb{C}$, $\forall z \in \mathbb{C}^2$

3) $G_F \circ F = d\, G_F$.

4) $G_{\lambda F} = G_F + \frac{1}{d-1}\log|\lambda|$ $\quad \forall \lambda \in \mathbb{C}^*$.

Proof: we need to show the uniform convergence of $G_n \to G_F$ on $\mathbb{C}^2\setminus\{0\}$.

From the previous theorem: $\frac{1}{C}\|F^n(z)\|^d \leq \|F^{n+1}(z)\| \leq C\|F^n(z)\|^d$.

take the log: $d\log\|F^n(z)\| - \log C \leq \log\|F^{n+1}(z)\| \leq d\log\|F^n(z)\| + \log C$.

divide by d^{n+1} and get $|G_{n+1}(z) - G_n(z)| \leq \frac{\log C}{d^{n+1}}$

By triangular inequality, $|G_{n+p}(z) - G_n(z)| \leq \log C \cdot \frac{1}{d^{n+1}}\left(1+\frac{1}{d}+\dots+\frac{1}{d^{p-1}}\right) < \log C \frac{1}{d^n}\frac{1}{d-1}$.

Hence the sequence (G_n) is uniformly Cauchy, and converges to a C^0 map G_F on $\mathbb{C}^2\setminus\{0\}$. (extend on 0 by $G_F(0) = -\infty$)

We also obtained (1) with $K = \frac{\log C}{d-1}$.

• $G_n(tz) = \frac{1}{d^n}\log\left(|t|^{d^n}\|F^n(z)\|\right) = \log|t| + G_n(z)$, and we get (2) when $n\to+\infty$

• $G_n(F(z)) = \frac{1}{d^n}\log\|F^{n+1}(z)\| = d\cdot G_{n+1}(z)$, and we get (3) when $n\to\infty$.

• $(G_n)_{\lambda F}(z) = \frac{1}{d^n}\cdot\log\|(\lambda F)^{\circ n}(z)\| = \frac{1}{d^n}\cdot\log|\lambda|^{\frac{d^n-1}{d-1}}\cdot\|F^{\circ n}(z)\| = \frac{1-d^{-n}}{d-1}\log|\lambda| + G_n(z)$
homogeneity

and we get (2) when $n\to+\infty$. $\square$

Rem: $\Omega_F = \{G_F < 0\}$, $\mathbb{C}^2\setminus\overline{\Omega_F} = \{G_F > 0\}$, $\partial\Omega_F = \{G_F = 0\}$.

Def: G_F is called the Green function associated to F.

• Plurisubharmonic functions:

Def: $U\subset\mathbb{C}^N$, $u: U\to\mathbb{R}\cup\{-\infty\}$ (u.s.c) is plurisubharmonic (psh) if $\forall L$ complex line, $u|_{U\cap L}$ is subharmonic

Equivalently:

Mean property on slices: $\forall a\in U$, $\xi\in\mathbb{C}^N$ s.t. $|\xi| < d(a, \mathbb{C}^N\setminus U)$, then:

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a+\xi e^{i\theta})\,d\theta.$$

Rem: sub harmonicity can be generalised to higher dimensions, and psh ⇒ sh

The properties seen for subharmonic functions extend to the case of psh functions

- existence of smoothing kernels
- limit of decreasing sequence of psh is psh
- u.s.c. regularisation of sup of psh is psh
- convex function evaluated on psh functions is psh

Ex: $f: U \subset \mathbb{C}^N \to \mathbb{C}$ holomorphic $\Rightarrow \log|f|$ is psh

if $u \in \mathcal{C}^2(U, \mathbb{R})$, u is psh $\Leftrightarrow$ $w \overset{u_\xi}{\mapsto} u(a + w\xi)$, $a \in U$, $\xi \in \mathbb{C}^N$ is s.h,

i.e. its Laplacian $\Delta u_\xi \geq 0$. We have

$$\frac{\partial^2}{\partial w \partial \bar{w}} u(a + w\xi) = \sum_{1 \leq j,k \leq N} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a + w\xi) \cdot \xi_j \cdot \bar{\xi}_k \geq 0$$

This is equivalent to asking that $\sum \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a)$ defines a semipositive Hermitian form at any $a \in U$.

This equivalence still holds for any psh function, in the sense of distributions.

Thm:

If $u : U \to \mathbb{R} \cup \{-\infty\}$ is psh, $u \not\equiv -\infty$ on every connected component of U

then $\forall \xi \in \mathbb{C}^N$, $Hu(\xi)$ defines a positive measure (by means of distributions)

$$Hu(\xi) = \sum_{1 \leq j,k \leq N} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k$$

(For $\xi = 0$ we get 0)

The data of a measure associated for any ξ, varying like this, corresponds to a $(1,1)$-current ($\sim$ $(1,1)$-form with distribution coefficients)

In fact, in this language, a psh function u gives rise to a current (positive $(1,1)$), $T = i\partial\bar{\partial}u$. Again, it comes from the formalism of forms.

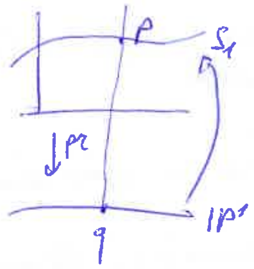
A map u is pluriharmonic $\overset{\text{def}}{\Leftrightarrow}$ u and $-u$ are psh $\Leftrightarrow \Delta u_\xi = 0 \ \forall \xi$

Theorem: Let F be a nondegenerate homogeneous polynomial map of degree $d\geq 2$, and G_F the associated Green function.
Let $f:\mathbb{P}^1 \circlearrowleft$ be the rational function induced by F, and $\mathrm{pr}:\mathbb{C}^2\setminus\{0\}\to\mathbb{P}^1$ the natural projection. Then the maximal open set in $\mathbb{C}^2\setminus\{0\}$ where G_F is pluriharmonic is $\mathrm{pr}^{-1}(F(f))$.

Proof: Call W such maximal open set.

$W \supseteq \mathrm{pr}^{-1}(F(f))$: let p_0 such that $\mathrm{pr}(p)\in F_f$.

Let s_1 be a local holomorphic section of pr defined on a nbhd of $q_0=\mathrm{pr}(p_0)$ and such that $s_1(q_0)=p_0$



$(t,q)\mapsto t\cdot s_1(q)$ gives a local system of coordinates of p.

and $G_F(t\, s_1(q)) = \log|t| + G_F(s_1(q))$.

we only have to show that $G_F\circ s_1$ is harmonic on a nbhd of q_0.

By hypothesis, $\exists\, V_1$ nbhd of q_0, and $\{f^{n_j}\}$ converging uniformly on V_1 to g.
We may assume: s_1 defined on V_1, $f^{n_j}(V_1)\subset V_2\ \forall j$, V_2 nbhd of $g(q_0)$ where there is a section s_2 of π_2.

Since $\mathrm{pr}\circ F^{n_j}\circ s_1 = f^{n_j} \Rightarrow F^{n_j}\circ s_1(q)=\lambda_j(q)\cdot s_2\circ f^{n_j}(q)$. for some $\lambda_j: V_1\to\mathbb{C}^*$ holomorphic.

$$\Rightarrow G_F\circ s_1(q) = \lim_{j\to\infty}\frac{1}{d^{n_j}}\log\|F^{n_j}\circ s_1(q)\| = \lim_{j\to\infty}\left(\frac{1}{d^{n_j}}\log|\lambda_j(q)| + \underbrace{\frac{1}{d^{n_j}}\log\|s_2\circ f^{n_j}(q)\|}_{\|s_2\circ g(q)\|\ \to\ 0}\right)$$

$$= \lim_{j\to\infty}\frac{1}{d^{n_j}}\log|\lambda_j(q)|$$

which is a uniform limit of harmonic functions $(\lambda_j(q)\neq 0)$

$\subseteq$: $p_0\in W \Rightarrow \exists\, h: U\to\mathbb{C}^*$ holomorphic defined on a nbhd U_0 of p in $\mathbb{C}^2\setminus\{0\}$ so that $G_F=\log|h|$

Then, $\left|\frac{1}{d^n}\log|h(z)|^{d^n} - \frac{1}{d^n}\log\|F^n(z)\|\right| \leq \frac{K}{d^n}\quad \forall z\in U_0$
(previous thm)
$G_F(z)$

Equivalently: $\left|\log\left\|F^n\left(\frac{z}{h(z)}\right)\right\|\right| \leq k \quad \forall z \in U_0$ (by homogeneity of F).

Let V_0 be a nbhd of $q_0 = pr(p_0)$, s a holomorphic section of pr on V_0 s.t. $s(V_0) \subset U_0$. Then $\tilde{s} := \frac{s}{h \circ s}$ is also a holomorphic section of pr on V_0, and $\{F^n \circ \tilde{s}\}$ is normal on W_0 (uniform boundedness).

Then $f^n = pr \circ F^n \circ \tilde{s}$ is also a normal family, and $pr(p_0) \in F_f$. $\square$

Rem: the proper formalism in the one of forms. In this case,

$\partial = dz = dx + i\,dy$, $\bar{\partial} = d\bar{z} = dx - i\,dy$, and

$\partial\bar{\partial} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\,dx \wedge dy \Rightarrow i\partial\bar{\partial} = i\,dz \wedge d\bar{z} = 2\,dx \wedge dy$ is a volume form.

The equilibrium measure

Prop: Let $F: \mathbb{C}^2 \to \mathbb{C}^2$ a non degenerate homogeneous polynomial map of degree $d \geq 2$, and G_F the associated Green function, f the induced map $f: \mathbb{P}^1 \circlearrowleft$.

Then there exists a positive measure μ_f on $\mathbb{P}^1$, so that, $\forall U \subset \mathbb{P}^1$ and all holomorphic sections $s: U \to \mathbb{C}^2 \setminus \{0\}$ of pr, we have $\mu_f|_U = i\partial\bar{\partial}(G_F \circ s)$

Proof: $\mu_f|_U$ is defined in the sense of distributions: $\forall \phi \in \mathcal{D}(U)$ (C^∞ with compact support inside U), $\langle \mu_f ; \phi \rangle = \frac{1}{\pi}\int (G_F \circ s) \cdot i\partial\bar{\partial}\phi$

normalization to have a probability measure; volume form

The definition does not depend on the section chosen:

If s_0, s_1 are two sections, then $\exists \lambda: U \to \mathbb{C}^*$ holomorphic, $s_0 = \lambda s_1$, and $\underbrace{i\partial\bar{\partial}(G_F \circ s_0)}_{\text{distrib.}} = \underbrace{i\partial\bar{\partial}\log|\lambda|}_{=0} + i\partial\bar{\partial}(G_F \circ s_1)$

because $\log|\lambda|$ is pluriharmonic

Similarly, it does not depend on the choice of the lift. $\square$

Def: μ_f is called the equilibrium measure of f, $\mathbb{P}^1 \circlearrowleft$.
or Green measure. ← take the one with total mass 1

Theorem: $f: \mathbb{P}^1 \circlearrowleft$, $\deg f = d \geq 2$, μ_f its equilibrium measure. Then:

(1) $\operatorname{supp} \mu_f = J_f$

(2) $f^* \mu_f = d \cdot \mu_f$; $f_* \mu_f = \mu_f$

Recall: $\cancel{f^* \mu_f(A)} =$ $\qquad f_* \mu_f(B) = \mu_f(f^{-1}(B))$

In fact, for $\phi \in \mathcal{D}(U)$, $f^*\phi = \phi \circ f$, $\langle f_* \mu ; \phi \rangle = \langle \mu, f^* \phi \rangle$.

For $f: \mathbb{P}^1 \circlearrowleft$ of degree d, $f_* \phi(y) = \sum_{x \in f^{-1}(y)} \phi(x)$, and $\langle f^* \mu, \phi \rangle := \langle \mu, f_* \phi \rangle$

(3) for any probability measure ν on $\mathbb{P}^1$ that does not charge $\mathcal{E}(f)$ ($\nu(\mathcal{E}(f)) = 0$), we have $\mu_f = \lim_{n \to \infty} \frac{1}{d^n} (f^n)^* \nu$.

In particular, $\forall a \notin \mathcal{E}(f)$, $\mu_f = \lim \frac{1}{d^n} \sum_{f^n(z) = a} \delta_z$
↑ counted with multiplicity.

Corollaries (already seen):

- $J_f \neq \emptyset$ (since $\mu_f > 0$ and $\operatorname{supp} \mu_f = J_f$)
- J_f is perfect: if $z \in J_f$, U nbhd, $U \setminus \{z\} \subset F_f \Rightarrow G_F \circ s$ is harmonic on $U \setminus \{z\}$ for any holo. section (G_F pluriharmonic exactly on $\pi^{-1}(F_f)$)
Being $G_F \circ s$ C^0, $\Rightarrow G_F \circ s$ is harmonic ~~on~~ U, and $U \cap \operatorname{Supp} \mu_f = \emptyset$, $z \notin J_f$.
- $\mathring{J}_f \neq \emptyset \Rightarrow J_f = \mathbb{P}^1$. ~~Set $U \subset \mathring{J}_f$ open, $\phi \in \mathcal{D}(U)$, ...~~

Properties: - μ_f is non atomic ...